Counting and Probability

Lesson 2 & 3 Permutations and Combinations

If there are 4 letters: R, G, B, P, how many ways are there to form a one letter string?

- ➔ There are 4 ways, the 4 possible strings are just "R", "G", "B", and "P"

How many ways are there to form a two letter string if each letter can only be used once?

- ➔ There are 4 choices for the first letter and 3 for the second, so there are $4 \cdot 3 = 12$ ways.

How many ways are there to form a two letter string?

- ➔ There are 4 choices for the first letter and 3 for the second, so there are $4 \cdot 3 = 12$ ways.

How many ways are there to form a three letter string?

- ➔ There are 4 choices for the first letter, 3 for the second, and 2 for the third, so there are $4 \cdot 3 \cdot 2 = 24$ ways.

How many ways are there to form a four letter string?

- ➔ There are 4 choices for the first letter, 3 for the second, 2 for the third, and one leftover for the last letter so there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways.

What about if each letter can be used more than once? How many options are there for a 3 letter word?

- ➔ There are 4 choices for the first letter, 4 for the second, and 4 for the third so there are $4 \cdot 4 \cdot 4 = 64$ ways.

What about if someone wants to form an ID in the form of an uppercase letter and then a number (including zero). For example: A7, Z0. How many possible IDs are there?

- ➔ There are 26 choices for the letter and then 10 for the number, so there are $26 \cdot 10 = 260$ total possible IDs.

How many 3 letter strings can be formed with the letters "A", "A", and "B"?

- ➔ There are $3 \cdot 2 \cdot 1 = 6$ possible strings, but some are repeated. For example, if we call the first A A1 and the second A A2, we would count A1A2B and A2A1B as separate strings, but they are actually the same: AAB. Thus, we must divide by 2.

How many 4 letter strings can be formed with the letters "A," "A," "A," and "B".

- ➔ There are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ possible strings, but some are repeated. For example, if we call the first A A1, the second A A2, and the third one A A3, we would count A1A2A3B, A1A3A2B, A2A1A3, A2A3A1, A3A1A2, A3A2A1 as separate strings, but they are actually the same: AAAB. Note that there are $3 \cdot 2 \cdot 1 = 6$ ways to arrange A1, A2, A3. Thus, we must divide by 6.

Say we want to choose a President, Vice President, and Secretary from a group of 10 students. A student can only hold one office. How many possible different combinations of president, vice president, and secretary can we have?

- ➔ There are 10 choices for president, 9 for VP, and 8 for secretary, so there are $10 \cdot 9 \cdot 8 = 720$ total combinations.

Now, say we want to know how many possible "officer groups" we can have. We do not care about who is president, VP, or secretary, we just care what 3 people are considered officers.

- ➔ Go back to the last problem. Say we have a given arrangement from the last problem: A is president, B is VP, C is secretary. Then, the officer group is just ABC. In the last problem, how many times did we count the officer group? We counted it 6 times, there are 3 choices of ABC for President, 2 choices for VP, and one for secretary. Thus, there are 720/6 = 120 possibilities.

Now, let us formalize what we just talked about.

To do this, we need to first define the factorial. The factorial is represented by an exclamation mark. $n! = n \cdot (n-1) \cdots 1$. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120.$ Similarly, 4! = $4 \cdot 3 \cdot 2 \cdot 1$. Note that $5! = 5 \cdot 4!$.

Side note: 0! = 1

What is $\frac{7!}{5!}$?

- ➔ $7! = 7 \cdot 6 \cdot 5!$. Thus, $\frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5!}{5!} = 7 \cdot 6 = 42$

Remember the string problem?

The premise of the problem was we have 4 letters and want to see how many different strings of a given length we can make. Letters are not allowed to be used more than once.

Length 1: 4
Length 2: $4 \cdot 3$

Length 3: $4 \cdot 3 \cdot 2$
Length 4: $4 \cdot 3 \cdot 2 \cdot 1$

Anyone see how this can be represented using factorials? Let the length be represented by l.

$\frac{4!}{(4-l)!}$ works!

In general, if you want to choose r elements from a group of n elements and order matters, the number of elements is $\frac{n!}{(n-r)!}$

Remember the officer group problem? First we found the number of possibilities where order matters (where we cared who was in what position) and then divided it by the number of ways to arrange a given group into those positions. We had a group of 3 people and wanted to know how to arrange them into 3 positions, so we did $3 \cdot 2 \cdot 1 = 3!$.

This is an example of a combination.

In general, if you want to choose r elements from a group of n elements and order doesn't matter, the number of elements is $\frac{n!}{r!(n-r)!}$

Guided Practice:

1. An outfit consists of a shirt, jacket, pants, and shoes. Alice owns 7 shirts, 4 jackets, 3 pants, and 2 shoes. How many possible outfits are there?
2. Alice has 6 different cans of cat food and wants to feed her cat one can per day. How many possible ways are there for her to feed her cat the cans over 4 days.
3. Alice is going to take a bag of snacks with her to school. She has 8 different snacks and wants to take 3 of them. How many different bags of snacks can she bring to school?
4. Alice has a cat and a dog. They each have their own separate treats. The cat has 5 different types of treats and the dog has 4 different types of treats. Assume there are an unlimited amount of treats of each type. How many possible combinations of treats can she give her dog and cat over 3 days?
5. How many permutations of "AABCDA" are there? A permutation is just rearranging the letters of the string.
6. A 5 person friend group is faced with the decision about whether to return to school or not. Each person has two options: they will either go to school

or they won't. How many possible groups are there of friends who go to school?

Homework:

1. A meal consists of a drink, appetizer, entree, and desert. There are 10 drink options, 5 appetizers, 4 entrees, and 5 deserts. How many different meals can you have?
2. Your dog refuses to eat a meal more than once. In order to get him to eat, you have tried to invent variations of meals. Each meal has 3 possible temperatures: warm, room temperature, and cold, 4 meat options, and 5 treat options. How many meals can you make?
3. You and 5 friends are sitting in a row of seats in the theater. How many different ways can you guys sit?
4. You and 7 friends (8 total) are on the school Disneyland trip. You will have 4 to a room. How many possible ways for you guys to divide into 2 rooms?
5. The teacher gives the class a movie day if the airplay code is of the form ABBA, such as 1221. Airplay codes are allowed to start with zeros and A and B may be the same. How many airplay codes will give you a movie day?
6. The teacher has amended his rules so he also gives you a movie day if the airplay code is of the form ABAB, such as 1212. How many total airplay codes give you a movie day? (Be careful not to overcount as codes such as 0000 count as both ABBAA and ABAB).